ON $k$-SEMI-PERFECT 1-FACTORIZATIONS OF $Q_n$ AND CRAFT’S CONJECTURE

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ABSTRACT. Let $n \geq 2$, $G$ be an $n$-regular graph and $1 \leq k \leq n-1$. An 1-factorization of the graph $G$ into 1-factors $F_1, \ldots, F_n$ shall be called $k$-semi-perfect, if $F_i \cup F_j$ forms a Hamiltonian cycle for every $1 \leq i \leq k$ and every $k+1 \leq j \leq n$.

The following results about the binary hypercube $Q_n$ are proved in this paper:

**Theorem 1.** (a) If $k = 1$ or $k = 5$ and $p \geq 1$ then there exists a $k$-semi-perfect 1-factorization of $Q_{k+2p}$.
(b) If $k \geq 1$ and $p \geq 1$ then there exists a $2k$-semi-perfect 1-factorization of $Q_{2k+2p}$.

**Theorem 2.** Let $n \geq 3$ and $F$ be a set of edges in $Q_n$. Assume also that either
(a) $n$ is odd and $0 \leq |F| \leq n-2$; or
(b) $n$ is even and $1 \leq |F| \leq n-2$.

Then there exist at least $k = n - |F| - 1$ Hamiltonian cycles in $Q_n - F$ that intersect on a perfect matching.

Also, a solution of Craft’s conjecture for the case $n = 6$ is provided and a conjecture that implies Craft’s conjecture for every $n \geq 5$ is formulated.

1. Introduction

Let $G$ be a simple graph. $G$ is decomposable into Hamiltonian cycles if there exist edge-disjoint Hamiltonian cycles which union covers all edges of $G$. A set $M$ of edges of $G$ is called matching if every vertex of $G$ is incident with at most one edge of $M$. A vertex $v$ of $G$ is covered by $M$ if $v$ is incident with an edge of $M$. A matching $M$ is called perfect if every vertex of $G$ is covered by $M$. Perfect matchings are also called 1-factors. A proper edge coloring of $G$ is an edge coloring for which every two edges with a common vertex have different colors. Clearly, if $G$ is $n$-regular (every vertex is incident to exactly $n$ edges) and properly colored with $n$ different colors, then every color class,

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i.e. all edges colored in one fixed color, is a perfect matching of \( G \), or equivalently, 1-factor. A proper edge coloring of an \( n \)-regular graph \( G \) with \( n \) colors is also called \( 1 \)-factorization. An 1-factorization is called \textit{perfect} if the union of any two 1-factors (color classes, perfect matchings) is a Hamiltonian cycle. An 1-factorization of \( G \) into 1-factors \( F_1, \ldots, F_n \) is called \textit{semi-perfect} if \( F_i \cup F_j \) forms a Hamiltonian cycle for any \( 1 < i < j < n \).

We extend the definition of semi-perfect 1-factorization as follows.

**Definition 1.1.** Let \( n \geq 2 \), \( G \) be an \( n \)-regular graph and \( 1 \leq k \leq n-1 \). An 1-factorization of the graph \( G \) into 1-factors \( F_1, \ldots, F_n \) shall be called \( k \)-semi-perfect, if \( F_i \cup F_j \) forms a Hamiltonian cycle for every \( 1 \leq i \leq k \) and every \( k+1 \leq j \leq n \).

It is clear from the definition that an 1-factorization is \( k \)-semi-perfect if and only if it is \((n-k)\)-semi-perfect. Also, in our terminology, the \( 1 \)-semi-perfect 1-factorizations (or, equivalently, the \((n-1)\)-semi-perfect 1-factorizations) are the well-known semi-perfect 1-factorizations.

The question of existence of perfect or semi-perfect 1-factorizations for different graphs \( G \) has been extensively studied (see [MR], [P], [K]). In this paper we study the existence of \( k \)-semi-perfect 1-factorizations for the \( n \)-regular graph \( Q_n \), where \( n \geq 2 \). We recall that the \( n \)-dimensional binary hypercube \( Q_n \) is the graph whose vertices are the binary sequences of length \( n \) and whose edges are pairs of binary sequences that differ in exactly one position. All edges that connect vertices that differ at a given position \( i \), \( 1 \leq i \leq n \), are called \textit{parallel} and define a \textit{direction} that we denote also by \( i \).

2. On Craft’s conjecture

In 1995 David Craft formulated the following conjecture (see [C]):

**Conjecture 2.1** (D. Craft). For each integer \( n \geq 2 \) there is a semi-perfect 1-factorization of \( Q_n \).

The following theorem settles Craft’s conjecture for odd \( n \). \(^1\)

**Theorem 2.2.** If \( n \geq 3 \) is odd then there exists 1-semi-perfect 1-factorization of \( Q_n \).

**Proof.** In what follows we shall use the following well-known fact: For any \( p \geq 1 \) the hypercube \( Q_{2p} \) is decomposable into \( p \) Hamiltonian cycles ([ABS], [B]).

\(^1\)We are grateful to Dan Archdeacon for helpful e-mail correspondence and to Giuseppe Mazzuoccolo who informed us, while we were writing this paper, that a proof of Craft’s conjecture for the case when \( n \) is odd appeared already in [KK].
Let \( n = 2p + 1 \geq 3 \). It is convenient to view the hypercube \( Q_{2p+1} \) as two copies \( Q_{2p}^0 \) and \( Q_{2p}^1 \) of the \( 2p \)-dimensional hypercube \( Q_{2p} \) such that every vertex \( v^0 \) in \( Q_{2p}^0 \) is connected by an edge to the vertex \( v^1 \) in \( Q_{2p}^1 \) which is the same binary sequence of length \( 2p \) as \( v^0 \). Clearly, the set \( M \) of all edges from \( Q_n \) that connect vertices from \( Q_{2p}^0 \) and \( Q_{2p}^1 \) form an 1-factor of \( Q_n \).

Let \( H_1, H_2, \ldots, H_p \) be a decomposition of \( Q_{2p} \) into \( p \) Hamiltonian cycles and let \( H_1^i, H_2^i, \ldots, H_p^i \) be the corresponding Hamiltonian decomposition of \( Q_{2p} \), \( i \in \{0, 1\} \). Every Hamiltonian cycle \( H_j, 1 \leq j \leq p \) defines in a natural way two 1-factors \( H_j(o) \) and \( H_j(e) \) in \( Q_{2p} \). (Starting from an edge, follow the cycle \( H_j \) (which has an even number of edges) and enumerate all edges. Then take for \( H_j(o) \) all odd-numbered edges and for \( H_j(e) \) all even-numbered edges.) Let \( H_1^i(x), H_2^i(x), \ldots, H_p^i(x), \) \( x \in \{o,e\}, i \in \{0,1\}, \) be the corresponding sets of 1-factors of \( Q_{2p}^i \), \( i \in \{0,1\} \) defined by \( H_1, H_2, \ldots, H_p \). Then

\[
\{ M, H_1^0(o) \cup H_1^1(e), H_1^0(e) \cup H_1^1(o), \ldots, H_p^0(o) \cup H_p^1(e), H_p^0(e) \cup H_p^1(o) \}
\]

form the required 1-semi-perfect 1-factorization of \( Q_n \) for the union of \( M \) and any other 1-factor forms a Hamiltonian cycle.

Semi-perfect 1-factorization of \( Q_3 \) and \( Q_4 \) were known to D. Craft. Using a computer program we were able to find all semi-perfect 1-factorizations of \( Q_4 \) and many different semi-perfect 1-factorizations of \( Q_6 \). One of those semi-perfect 1-factorizations of \( Q_6 \) is given in Appendix A, where each \( F_i \), \( 1 \leq i \leq 6 \), is a perfect matching of \( Q_6 \) and \( F_2 \cup F_3 \), \( 2 \leq i \leq 6 \), is a Hamiltonian cycle of \( Q_6 \). Since all edges in \( F_i \) are parallel to one direction, it easily follows that the projections parallel to that direction of all edges from \( F_i \), \( 2 \leq i \leq 6 \), onto \( Q_5 \), form a Hamiltonian cycle of \( Q_5 \). Therefore there exist two partitions of the set of edges of \( Q_5 \) into perfect matchings \( \{M_1, M_2, \ldots, M_5\} \) and \( \{N_1, N_2, \ldots, N_5\} \) such that \( M_i \cup N_i \) forms a Hamiltonian cycle for each \( 1 \leq i \leq 5 \). It is interesting to note that a semi-perfect 1-factorization of \( Q_4 \) with a perfect matching, all edges of which are parallel to one direction, does not exist; or equivalently, two partitions of the set of edges of \( Q_3 \) into perfect matchings \( \{M_1, M_2, M_3\} \) and \( \{N_1, N_2, N_3\} \) such that \( M_i \cup N_i \) forms a Hamiltonian cycle for each \( 1 \leq i \leq 3 \), do not exist.

Since for any \( p \geq 1 \) the hypercube \( Q_{2p} \) is decomposable into \( p \) Hamiltonian cycles, it easily follows that for every even \( n \geq 2 \) there exist two partitions of the set of edges of \( Q_n \) into perfect matchings \( \{M_1, M_2, \ldots, M_n\} \) and \( \{N_1, N_2, \ldots, N_n\} \) such that \( M_i \cup N_i \) forms a
Hamiltonian cycle for each $1 \leq i \leq n$. Based on the above observations we state the following conjecture.

**Conjecture 2.3.** Let $n = 2$ or $n \geq 4$. Then there exist two 1-factorizations $\{M_1, M_2, \ldots, M_n\}$ and $\{N_1, N_2, \ldots, N_n\}$ of $Q_n$ such that $M_i \cup N_i$ forms a Hamiltonian cycle for each $1 \leq i \leq n$.

We emphasize again that the above conjecture is a theorem for every even positive integer $n$ and for $n = 5$, but it is not true for $n = 3$. Also, Craft’s conjecture (for $n \geq 5$) is an easy corollary from it.

3. On $k$-semi-perfect 1-factorizations of $Q_n$

Another corollary of Conjecture 2.3 is the following.

**Theorem 3.1.** Let $k \geq 2$, $p \geq 1$ be two integers and let Conjecture 2.3 hold true for $k$. Then there exists a $k$-semi-perfect 1-factorization of $Q_{k+2p}$.

**Proof.** It is convenient to view the $n$-dimensional hypercube $Q_{k+2p}$ as a $2p$-dimensional hypercube $Q_{2p}$ which “vertices” are $k$-dimensional hypercubes $Q_k$ (i.e. we view $Q_{k+2p}$ as a Cartesian product of $Q_{2p}$ and $Q_k$). We enumerate all vertices of $Q_{2p}$ from 1 to $2^{2p}$ (we take the binary representation of each vertex plus one) and we denote by $Q_k$, $1 \leq i \leq 2^{2p}$, the $k$-dimensional hypercube which is in the “vertex” of $Q_{2p}$ numbered $i$. Also, we enumerate all vertices of $Q_k$ from 1 to $2^k$ (the binary representation of each vertex plus one) and for each $1 \leq i \leq 2^{2p}$ we take that same enumeration of the vertices in $Q_k$.

Let $H_1, H_2, \ldots, H_p$ be a decomposition of $Q_{2p}$ into $p$ Hamiltonian cycles. For convenience, we enumerate the edges of each such Hamiltonian cycle starting from an edge that contains the vertex numbered 1. Then every Hamiltonian cycle $H_j$, $1 \leq j \leq p$ defines in a natural way two 1-factors $H_j(o)$ and $H_j(e)$ in $Q_{2p}$. Let $\{H_1(x), H_2(x), \ldots, H_p(x)\}$, $x \in \{o, e\}$, be the corresponding two sets of 1-factors of $Q_{2p}$ defined by $H_1, H_2, \ldots, H_p$.

Let also $\{L_1(x), L_2(x), \ldots, L_k(x)\}$, $x \in \{o, e\}$ be two 1-factorizations of $Q_k$ such that $L_r := L_r(e) \cup L_r(o)$ is a Hamiltonian cycle for each $1 \leq r \leq k$ and let $\{L_1^i(x), L_2^i(x), \ldots, L_k^i(x)\}$, $x \in \{o, e\}$, be the corresponding two sets of 1-factors of $Q_k$, $1 \leq i \leq 2^{2p}$, defined by $L_1(x), L_2(x), \ldots, L_k(x)$.

We define the first set of $k$ 1-factors of $Q_{k+2p}$ in the following way:

$$M_r := L_r^1(e) \cup \bigcup_{2 \leq i \leq 2^{2p}} L_r^i(o), 1 \leq r \leq k.$$  

Since $Q_{2p}$ is a bipartite graph, and according to our enumeration of the edges, each edge in $H_i(e)$, following the cycle $H_i$, connects even
numbered vertices \( i_1 \) in \( Q_{2p} \) (that could be considered as vertices in \( Q_k \)) to odd numbered vertices \( i_2 \) in \( Q_{2p} \) (that could be considered as vertices in \( Q_k \)), and that each edge in \( H_t(o) \), following the cycle \( H_t \), connects odd numbered vertices \( i_1 \) in \( Q_{2p} \) to even numbered vertices \( i_2 \) in \( Q_{2p} \). Notice also that for each \( j \), \( 1 \leq j \leq p \), each edge in \( H_j(e) \) and \( H_j(o) \) corresponds to \( 2^k \) edges in \( Q_n \). Let \( H_j(x,y) \) be the set of all edges in \( Q_n \) corresponding to such edges from \( H_j(x) \) that, following the cycle \( H_j \), begin from a vertex with parity \( y \) in \( Q_k \) whenever \( i \) has parity \( x \), where \( 1 \leq i \leq 2^p \).

Now we define the second set of \( 2p \) 1-factors in the following way:

\[
N_j(y) := H_j(e, y) \cup H_j(o, y), \quad 1 \leq j \leq p, \quad y \in \{e, o\}.
\]

It follows from the definitions that

\[
\mathcal{F} := \bigcup \{M_r : 1 \leq r \leq k\} \cup \bigcup \{\{N_j(e), N_j(o)\} : 1 \leq j \leq p\}
\]

is a set of \( k + 2p \) pairwise disjoint 1-factors. To finish the proof we need to show that \( \mathcal{F} \) is a \( k \)-semi-perfect 1-factorization for \( Q_{k+2p} \), i.e. that \( M_r \cup N_j(y) \) is a Hamiltonian cycle for each \( 1 \leq r \leq k \), each \( y \in \{e, o\} \) and each \( 1 \leq j \leq p \).

We fix \( 1 \leq r \leq k \), \( y \in \{e, o\} \) and \( 1 \leq j \leq p \). We shall show that \( C := M_r \cup N_j(y) \) is a Hamiltonian cycle.

Below if \( v \) is a vertex in \( Q_k \) numbered \( s \) then the corresponding vertex in \( Q_k \) is denoted by \( s(i) \).

Let \( 1 = s_1, s_2, \ldots, s_{2k} \) be the sequence of numbers of the vertices in \( Q_k \) beginning from 1 that represents the cycle \( L_r \). We can arrange (by changing the direction, if necessary) that the edge \((s_1(1), s_2(1))\) does not belong to \( C \). Let \( 1 = q_1, q_2, \ldots, q_{2^p} \) be a sequence of numbers of the vertices in \( Q_{2p} \) beginning from 1 that represents the cycle \( H_j \). If the edge \((s_1(1), s_1(q_2))\) belongs to \( C \), then the following vertices belong to \( C \) and the edges between them form a path:

\[
s_1(1), s_1(q_2), s_2(q_2), s_2(q_3), s_1(q_3), \ldots, s_1(q_{2^p}), s_2(q_{2^p}), s_2(1), s_3(1).
\]

The length of this path is \( 2^{2p+1} \) and it contains all vertices of the type \( s_1(q_i) \) and \( s_2(g_i) \), where \( 1 \leq i \leq 2^p \), and connects \( s_1(1) \) with \( s_3(1) \). Using similar paths we can connect \( s_3(1) \) with \( s_5(1) \), and so on, \( s_{2k-1}(1) \) with \( s_1(1) \). There are \( 2^{k-1} \) such paths which are edge disjoint. All these paths form a Hamiltonian cycle of \( Q_{k+2p} \) that coincides with \( C \).

In a similar way, if the edge \((s_1(1), s_1(q_2))\) does not belong to \( C \), then \((s_2(1), s_2(q_2))\) belongs to \( C \). Then the following vertices belong to \( C \) and the edges between them form a path:

\[
s_2(1), s_2(q_2), s_1(q_2), s_1(q_3), s_2(q_3), \ldots, s_2(q_{2^p}), s_1(q_{2^p}), s_1(1), s_{2k}(1).
\]
The length of this path is $2^{2p+1}$ and it contains all vertices of the type $s_1(q_i)$ and $s_2(q_i)$, where $1 \leq i \leq 2^{2p}$, and connects $s_2(1)$ with $s_{2k}(1)$. Using similar paths we can connect $s_{2k}(1)$ with $s_{2k-1}(1)$, and so on, $s_4(1)$ with $s_2(1)$. There are $2^{k-1}$ such paths which are edge disjoint. All these paths form a Hamiltonian cycle of $Q_{k+2p}$ that coincides with $C$. □

As a corollary of the above theorem we obtain the following.

**Corollary 3.2.** (a) If $n \geq 4$ is even then for every even $k$, $2 \leq k \leq n-2$, there exists a $k$-semi-perfect 1-factorization of $Q_n$.

(b) If $p \geq 1$ then there exists a 5-semi-perfect 1-factorization of $Q_{5+2p}$.

Using (a) of the above corollary, Theorem 2.2 and some observations from the proof of Theorem 3.1 we prove the following.

**Theorem 3.3.** Let $n \geq 3$ and $F$ be a set of edges in $Q_n$. Assume also that either

(a) $n$ is odd and $0 \leq |F| \leq n-2$; or

(b) $n$ is even and $1 \leq |F| \leq n-2$.

Then there exist at least $k = n - |F| - 1$ Hamiltonian cycles in $Q_n - F$ that intersect on a perfect matching.

**Proof.** (a) Let $n \geq 3$ be odd. Then, according to Theorem 2.2, there exists 1-semi-perfect 1-factorization of $Q_n$. From its proof we know that one of the 1-factors (say $M$) could be chosen such that all of its edges to be parallel to a chosen direction. We choose a direction in which there are no deleted edges (clearly such a direction exists). Since there are only $|F|$ deleted edges, at least $k = n - |F| - 1 \geq 1$ of the remaining 1-factors do not contain deleted edges. Each one of these 1-factors together with $M$ forms a Hamiltonian cycle of $Q_n - F$. Therefore there are at least $k$ Hamiltonian cycles of $Q_n - F$ that intersect on the perfect matching $M$.

(b) Let $n \geq 4$ be even. Since $1 \leq |F| \leq n-2$ there exists at least one direction $i_1$ such that there is an edge in $F$ parallel to that direction. Also, there exists a direction $i_2$ such that no edge in $F$ is parallel to that direction. Now, as in the proof of Theorem 3.1, we view the $n$-dimensional hypercube $Q_n$ as an $n-2$-dimensional hypercube $Q_{n-2}$ which “vertices” are 2-dimensional hypercubes $Q_2$ (i.e. we view $Q_n$ as a Cartesian product of $Q_{n-2}$ and $Q_2$). Clearly, we can arrange $i_1$ and $i_2$ to belong to $Q_2$. Then, according to Theorem 3.1, there exists a 2-semi-perfect 1-factorization of $Q_n$. Since $2^{n-2} > n-2$ there exists a vertex in $Q_{n-2}$ such that no edge from $F$ belongs to its “vertex” $Q_2$. 


Therefore, if in the proof of Theorem 3.1 we begin the enumeration of the vertices of $Q_{n-2}$ from that particular vertex then one of the two 1-factors

$$M(x) := L^1(x) \cup \bigcup_{2 \leq i \leq 2^{n-2}} L^i(x), \; x \in \{e, o\}.$$  

will not contain edges from $F$. Denote that 1-factor by $M$. Since there are only $|\mathcal{F}|$ deleted edges, at least $k = n - |\mathcal{F}| - 1 \geq 1$ of the $n - 2$ 1-factors defined in the proof of Theorem 3.1 by

$$N_j(y) := H_j(e, y) \cup H_j(o, y), \; 1 \leq j \leq \frac{n-2}{2}, \; y \in \{e, o\}$$

do not contain deleted edges. Each one of these 1-factors together with $M$ forms a Hamiltonian cycle of $Q_n - \mathcal{F}$. Therefore there are at least $k$ Hamiltonian cycles of $Q_n - \mathcal{F}$ that intersect on the perfect matching $M$.

As a direct corollary of Theorem 3.3 we obtain the following result (see [LZB] and [SSB]).

**Corollary 3.4.** Let $n \geq 2$ and $\mathcal{F}$ be a set of up to $n - 2$ edges in $Q_n$. Then $Q_n - \mathcal{F}$ is Hamiltonian.

### Appendix A. Solution of Craft’s conjecture for $n = 6$

Below we provide one of the solutions of Craft’s conjecture for $n = 6$. Since $F_1, \ldots, F_6$ form an 1-factorization of $Q_6$, the enumeration of the vertices of $Q_6$ from 1 to 64 should be clear. The 1-factor that forms a Hamiltonian cycle with any other 1-factor is $F_1$.

- $F_1 = \{(1, 17), (2, 18), (3, 19), (4, 20), (5, 21), (6, 22), (7, 23), (8, 24), (9, 25), (10, 26), (11, 27), (12, 28), (13, 29), (14, 30), (15, 31), (16, 32), (33, 49), (34, 50), (35, 51), (36, 52), (37, 53), (38, 54), (39, 55), (40, 56), (41, 57), (42, 58), (43, 59), (44, 60), (45, 61), (46, 62), (47, 63), (48, 64)\};

- $F_2 = \{(1, 2), (3, 4), (5, 6), (7, 8), (9, 10), (11, 12), (13, 14), (15, 16), (17, 19), (18, 24), (20, 22), (21, 29), (23, 39), (25, 27), (26, 32), (28, 30), (31, 47), (33, 35), (34, 42), (36, 44), (37, 45), (38, 40), (41, 43), (46, 48), (49, 50), (51, 53), (52, 54), (55, 56), (57, 58), (59, 60), (61, 62), (63, 64)\};$

- $F_3 = \{(1, 3), (2, 4), (5, 7), (6, 8), (9, 11), (10, 12), (13, 15), (14, 16), (17, 18), (19, 21), (20, 36), (22, 38), (23, 24), (25, 31), (26, 42), (27, 28), (29, 30), (32, 48), (33, 41), (34, 40), (35, 37), (39, 47), (43, 44), (45, 46), (49, 51), (50, 52), (53, 55), (54, 62), (56, 64), (57, 59), (58, 60), (61, 63)\};$

- $F_4 = \{(1, 7), (2, 8), (3, 5), (4, 6), (9, 15), (10, 16), (11, 13), (12, 14), (17, 33), (18, 20), (19, 35), (21, 22), (23, 31), (24, 40), (25, 26), (27, 43), (28, 44), (29, 45), (30, 32), (34, 36), (37, 39), (38, 46), (41, 42), (47, 48), (49, 55), (50, 58), (51, 59), (52, 60), (53, 61), (54, 56), (57, 63), (62, 64)\};$
\[ F_5 = \{(1, 9), (2, 10), (3, 11), (4, 12), (5, 13), (6, 14), (7, 15), (8, 16), (17, 23), (18, 34), (19, 20), (21, 37), (22, 24), (25, 41), (26, 28), (27, 29), (30, 46), (31, 32), (33, 39), (35, 43), (36, 38), (40, 48), (42, 44), (45, 47), (49, 57), (50, 56), (51, 52), (53, 54), (55, 63), (58, 64), (59, 61), (60, 62)\}; \\
F_6 = \{(1, 49), (2, 50), (3, 51), (4, 52), (5, 53), (6, 54), (7, 55), (8, 56), (9, 57), (10, 58), (11, 59), (12, 60), (13, 61), (14, 62), (15, 63), (16, 64), (17, 25), (18, 26), (19, 27), (20, 28), (21, 23), (22, 30), (24, 32), (29, 31), (33, 34), (35, 36), (37, 38), (39, 40), (41, 47), (42, 48), (43, 45), (44, 46)\}. \\

References


[C] Dan Archdeacon: Problems in Topological Graph Theory http://www.cems.uvm.edu/~archdeac/problems/perfectq.htm


